

# Maximal Strategies for Paramodulation with Non-Monotonic Orderings

Miquel Bofill and Guillem Godoy

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## Abstract

A *west ordering* is a well-founded (strict partial) ordering on terms that satisfies the subterm property. In [BGNR99] the completeness of an ordered paramodulation inference system w.r.t. west orderings was proved, thus dropping for the first time the monotonicity requirements on the ordering. However, the inference system still required the eager *selection* of negative equations. Here we improve upon [BGNR99] in two directions.

On the one hand, we show that the results are compatible with constraint inheritance and the so-called *basic* strategy [BGLS95, NR95], thus further restricting the search space.

On the other hand, we introduce an inference system where also the positive equations of non-unit clauses can be selected, provided that they are maximal.

## 1 Introduction

Deduction with equality is fundamental in mathematics, logics and many applications of formal methods in computer science. During the last two decades this field has importantly progressed through new Knuth-Bendix-like completion techniques and their extensions to ordered paramodulation for first-order clauses. These techniques have lead to important results on theorem proving in first-order logic with equality [HR91, BDH86, BD94, BG94] (that have been applied to state-of-the-art theorem provers like Spass [Wei97]), results on logic-based complexity and decidability analysis [BG96, Nie98], on deduction with *constrained clauses* [KKR90, NR95], and on many other applications like inductive theorem proving, symbolic constraint solving, or equational-logic programming.

But, until very recently, all completeness results for Knuth-Bendix completion and ordered paramodulation required the term ordering  $\succ$  to be well-founded, monotonic and total (or extendable to a total ordering) on ground terms. All main proof techniques, like the transfinite semantic tree method [HR91], the proof ordering method [BDH86, BD94], and the model generation method [BG94] relied at some point on these requirements. Moreover, in many practical situations these requirements are too strong.

For example, one may want to apply completion to a terminating set of rules with a given orientation by a reduction ordering that cannot be extended to a total one, like  $f(a) \rightarrow f(b)$  and  $g(b) \rightarrow g(a)$ , for which  $a$  and  $b$  must be uncomparable in any monotonic extension.

Another typical situation is deduction modulo built-in equational theories  $E$ , where the existence of a total *E-compatible* reduction ordering is a very strong requirement. For example, the existence of such an ordering for the case where  $E$  consists of associativity and commutativity (AC) properties for some symbols remained open for a long time, and, once it was found, it triggered quite a number of results, like the decidability of the ground AC-word and -unification problems. Unfortunately, for many  $E$  such orderings cannot exist. For instance, when  $E$  contains an idempotency axiom  $f(x, x) = x$ , then if  $s \succ t$ , by monotonicity one should have  $f(s, s) \succ f(s, t)$ , which by  $E$ -compatibility implies  $s \succ f(s, t)$  and hence non-well-foundedness.

Recently, in [BGNR99] we introduced techniques for dropping the monotonicity requirement that open the door to deduction modulo many more classes of equational theories. The only properties required for  $\succ$  are well-foundedness and the subterm property. Our technique is a variant of the model generation technique, with the main difference that the termination of the ground rewrite system  $R$  that defines the model is not a consequence of the ordering, and hence termination is proved otherwise. Induction on an extension of the reduction ordering induced by  $R$  is then used for proving the main completeness theorem.

However, the inference system of [BGNR99] still required the eager *selection* of negative equations. Here we improve upon [BGNR99] in two directions.

On the one hand, in Section 2 we show that the results are compatible with constraint inheritance and hence with the *basic* strategy [BGLS95, NR95], thus further restricting the search space.

On the other hand, in Section 3 we introduce an inference system where also the positive equations of non-unit clauses can be selected, provided that they are maximal, and we prove the corresponding completeness result by means of proof transformations.

## 1.1 Preliminaries

We use the standard definitions of [DJ90]:  $T(\mathcal{F}, \mathcal{X})$  ( $T(\mathcal{F})$ ) is the set of (ground) terms over  $\mathcal{F}$ , the subterm of  $t$  at *position*  $p$  is denoted  $t|_p$ , the result of replacing  $t|_p$  by  $s$  in  $t$  is denoted  $t[s]_p$ , and syntactic equality of terms is denoted by  $\equiv$ .

If  $\rightarrow$  is binary relation, then  $\leftarrow$  is its inverse,  $\leftrightarrow$  is its symmetric closure,  $\rightarrow^+$  is its transitive closure and  $\rightarrow^*$  is its reflexive-transitive closure. We write  $s \rightarrow^! t$  if  $s \rightarrow^* t$  and there is no  $t'$  such that  $t \rightarrow t'$ . Then  $t$  is called *irreducible* and a *normal form* of  $s$  (w.r.t.  $\rightarrow$ ). The relation  $\rightarrow$  is *well-founded* or *terminating* if there exists no infinite sequence  $s_1 \rightarrow s_2 \rightarrow \dots$  and it is *confluent* or *Church-Rosser* if the relation  $\leftarrow^* \circ \rightarrow^*$  is contained in  $\rightarrow^* \circ \leftarrow^*$ . It is *locally confluent* if  $\leftarrow \circ \rightarrow \subseteq \rightarrow^* \circ \leftarrow^*$ . By Newman's lemma, terminating locally-confluent relations are confluent. A relation  $\rightarrow$  on terms is *monotonic* if  $s \rightarrow t$  implies  $u[s]_p \rightarrow u[t]_p$  for all terms  $s, t$  and  $u$  and positions  $p$ . A *congruence* is a reflexive, symmetric, transitive and monotonic relation on terms.

An *equation* is a multiset  $\{s, t\}$ , denoted  $s \simeq t$  or, equivalently,  $t \simeq s$ . A first-order clause is a pair of finite multisets of equations  $\Gamma$  (the *antecedent*) and  $\Delta$  (the *succedent*), denoted by  $\Gamma \rightarrow \Delta$ . The empty clause  $\square$  is a clause where both  $\Gamma$  and  $\Delta$  are empty.

A *rewrite rule* is an ordered pair of terms  $(s, t)$ , written  $s \rightarrow t$ , and a set of rewrite rules  $R$  is a *term rewrite system* (TRS). The rewrite relation with  $R$  on  $T(\mathcal{F}, \mathcal{X})$ , denoted  $\rightarrow_R$ , is the smallest monotonic relation such that  $l\sigma \rightarrow_R r\sigma$  for all  $l \rightarrow r \in R$  and all  $\sigma$ , and if  $s \rightarrow_R t$  then we say that  $s$  *rewrites into*  $t$  with  $R$ .  $R$  is called terminating, confluent, etc.

if  $\rightarrow_R$  is. A rewrite system  $R$  is *convergent* if it is confluent and terminating; then every term  $t$  has a unique normal form w.r.t.  $\rightarrow_R$ , denoted by  $nf_R(t)$ , and  $s \simeq t$  is a logical consequence of  $R$  (where  $R$  is seen as a set of equations) iff  $nf_R(s) = nf_R(t)$ .

Let  $R$  be a set of ground equations or rewrite rules. Then the congruence  $\leftrightarrow_R^*$  defines an equality Herbrand interpretation denoted by  $R^*$ , where the only predicate  $\simeq$  is interpreted by  $s \simeq t$  iff  $s \leftrightarrow_R^* t$ . We write  $s = t \in R^*$  if  $s \leftrightarrow_R^* t$ .  $R^*$  satisfies (is a model of) a ground clause  $\Gamma \rightarrow \Delta$ , denoted  $R^* \models \Gamma \rightarrow \Delta$ , if  $R^* \not\models \Gamma$  or  $R^* \cap \Delta \neq \emptyset$ . The empty clause  $\square$  is hence satisfied by no interpretation.  $R^*$  satisfies a set of clauses  $S$ , denoted by  $R^* \models S$ , if it satisfies every clause in  $S$ . For dealing with non-equality predicates, atoms  $A$  can be expressed by equations  $A \simeq \text{true}$  where  $\text{true}$  is a new symbol.

## 1.2 Some properties of ground TRS and orderings

A (strict partial) *ordering* on  $T(\mathcal{F}, \mathcal{X})$  is an irreflexive transitive relation  $\succ$ . It is a *reduction ordering* if it is well-founded and monotonic, and stable under substitutions:  $s \succ t$  implies  $s\sigma \succ t\sigma$  for all substitutions  $\sigma$ . It fulfils the *subterm property* if  $\succ \supseteq \triangleright$ , where  $\triangleright$  denotes the strict subterm ordering.

**Definition 1** A west-ordering is a well-founded ordering on  $T(\mathcal{F})$  that fulfils the subterm property and that is total on  $T(\mathcal{F})$  (it is called west after well-founded, subterm and total).

Not all well-founded orderings on terms can be extended to west-orderings, even if they do not contradict the subterm property. For example, if  $a \succ_1 f(b)$  and  $b \succ_1 f(a)$ , then, if  $\succ$  is  $(\succ_1 \cup \triangleright)^+$ , we get  $a \succ f(b) \succ b \succ f(a) \succ a$ . But every well-founded ordering can be totalized [Wec91], and hence every well-founded ordering satisfying the subterm property can be extended to a west ordering. We also have the following:

**Lemma 2** Every reduction ordering  $\succ_r$  can be extended to a west ordering.

**Proof:** Let  $\succ_{rs}$  be  $(\succ_r \cup \triangleright)^+$ . Then  $\succ_{rs}$  is well-founded: let  $s_1 \succ_{rs} s_2 \succ_{rs} \dots$  be an infinite sequence with  $s_1$  minimal w.r.t.  $\succ_r$ ; this is impossible since by monotonicity of  $\succ_r$   $s[t] \triangleright t \succ_r u$  implies  $s[t] \succ_r s[u] \triangleright u$  and hence the sequence can be re-arranged such that  $s_1 \succ_r s'_2$  for some  $s'_2$ .  $\square$

**Lemma 3** Let  $R$  be a ground TRS such that for all rules  $l \rightarrow r$  in  $R$  the term  $r$  is irreducible by  $R$ . Then  $R$  is terminating.

**Proof:** Assume  $R$  is non-terminating. Then there exists an infinite rewrite sequence  $t_1 \rightarrow_R t_2 \rightarrow_R \dots$ . It is easy to extract an infinite subsequence  $s_1 \rightarrow_R s_2 \rightarrow_R \dots$  of it where there is at least one rewrite step  $s_i \rightarrow_R s_{i+1}$  at the topmost position, i.e., where  $s_i \equiv l$  and  $s_{i+1} \equiv r$  for some rule  $l \rightarrow r$  in  $R$ . But then  $s_{i+1}$  is irreducible by  $R$ , contradicting the infiniteness assumption.  $\square$

## 2 Completeness of the positive unit strategy with equality and ordering constraint inheritance

We show here the refutational completeness of our inference system for the case of Horn clauses. We assume that all clauses with negative equations have one of them selected arbitrarily. The inference system  $\mathcal{J}$  for Horn clauses with equality is (selected equations are written underlined>):

*paramodulation right:*

$$\frac{\rightarrow l \simeq r \mid T_1 \quad \rightarrow s \simeq t \mid T_2}{\rightarrow s[r]_p \simeq t \mid s|_p = l \wedge l > r \wedge T_1 \wedge T_2} \quad \text{if } s|_p \notin \mathcal{X}$$

*paramodulation left:*

$$\frac{\rightarrow l \simeq r \mid T_1 \quad \Gamma, \underline{s \simeq t} \rightarrow \Delta \mid T_2}{\Gamma, s[r]_p \simeq t \rightarrow \Delta \mid s|_p = l \wedge l > r \wedge T_1 \wedge T_2} \quad \text{if } s|_p \notin \mathcal{X}$$

*equality resolution:*

$$\frac{\Gamma, \underline{s \simeq t} \rightarrow \Delta \mid T}{\Gamma \rightarrow \Delta \mid s = t \wedge T}$$

where  $=$  and  $>$  are interpreted respectively as the syntactic equality relation  $\equiv$  and the given west ordering  $\succ$  when dealing with instances. That is, we forbid those instances of the conclusion that correspond to ground inferences between instances of the premises for which the constraints do not hold.

We call a set of constrained Horn clauses  $S$  *closed under  $\mathcal{J}$  with equality and ordering constraint inheritance* if  $D \mid s = t \wedge OC \wedge T_1 \wedge \dots \wedge T_n$  is in  $S$  whenever  $C_1 \mid T_1, \dots, C_n \mid T_n$  are clauses in  $S$  and there is an inference by  $\mathcal{J}$  with premises  $C_1, \dots, C_n$  and conclusion  $D \mid s = t \wedge OC$  and  $s = t \wedge OC \wedge T_1 \wedge \dots \wedge T_n$  is satisfiable.

**Definition 4** *Given an ordering  $\succ$ , we define  $\succ_{mul}$  to be the smallest ordering on multisets such that:*

$$M \cup \{s\} \succ_{mul} N \cup \{t_1, \dots, t_n\} \text{ if } M = N \text{ and } s \succ t_i \text{ for all } i \in 1 \dots n.$$

If  $\succ$  is well founded and total on  $S$ , so is  $\succ_{mul}$  on finite multisets over  $S$  [DM79].

This will be used to lift orderings  $\succ$  on terms to orderings on equations and clauses:

**Definition 5** *Let  $C$  be a ground clause, and let  $emul(s \simeq t)$  be  $\{s, t\}$  if  $s \simeq t$  is a positive equation in  $C$ , and  $\{s, s, t, t\}$  if it is negative. Then, if  $\succ$  is an ordering, we define the ordering  $\succ_e$  on (occurrences of) ground equations in a clause by  $e \succ_e e'$  if  $emul(e) \succ_{mul} emul(e')$ . Similarly,  $\succ_c$  on ground clauses is defined  $C \succ_c D$  if  $mse(C) (\succ_{mul})_{mul} mse(D)$ , where  $mse(C)$  is the multiset of all  $emul(e)$  for occurrences  $e$  of equations in  $C$ .*

**Definition 6** *Let  $S$  be a set of clauses. An instance  $C$  of the form  $\rightarrow l \simeq r$  of a clause in  $S$  generates the rule  $l \rightarrow r$  if*

- (i)  $R_C^* \not\models C$ ,
- (ii)  $l \succ r$  and
- (iii)  $l$  and  $r$  are irreducible by  $R_C$ .

where  $R_C$  is the set of rules generated by all instances  $D$  of clauses in  $S$  such that  $C \succ_c D$ . We denote by  $R_S$  the set of rules generated by all ground instances of  $S$ .

**Lemma 7** *Let  $S$  be a set of clauses. Then the ground TRS  $R_S$  is convergent.*

**Proof:** For termination, by Lemma 3 it suffices to show that for every rule  $l \rightarrow r$  in  $R_S$  the term  $r$  is irreducible by  $R_S$ . By construction, if an instance  $C$  generates  $l \rightarrow r$ , the term  $r$  is irreducible by  $R_C$ . Since  $l \succ r$ , and  $\succ$  fulfils the subterm property, clearly  $l \rightarrow r$  itself does not reduce  $r$  either. Finally, if  $l' \rightarrow r'$  is generated by an instance  $D$  with  $D \succ_c C$  then, by definition of  $\succ_c$ , we must have  $l' \succ l$  and hence  $l'$  cannot be a subterm of  $r$  either. This proves termination.

For similar reasons  $l$  is irreducible by  $R_S \setminus \{l \rightarrow r\}$ . This means that  $\rightarrow_{R_S}$  is locally confluent and hence, since it is terminating, confluent.  $\square$

**Definition 8** *Let  $S$  be a set of clauses. By  $\succ_R$  we denote the ordering  $\rightarrow_{R_S}^+$ .*

**Lemma 9** *Let  $S$  be a set of clauses. Then  $\succ_R$  is a reduction ordering.*

**Proof:** Since  $R_S$  is a terminating TRS,  $\rightarrow_{R_S}^+$  is a reduction ordering. [DJ90].  $\square$

**Theorem 10** *The inference system  $\mathcal{J}$  with equality and ordering constraint inheritance is refutationally complete for Horn clauses with variables.*

**Proof:** Let  $S_0$  be a set of —unconstrained— Horn clauses, and let  $S$  be the closure of  $S_0$  under  $\mathcal{J}$  with equality and ordering constraint inheritance. We show that  $R_S^*$  is a model for  $S$  whenever  $\square \notin S$ .

We first prove that  $R_S^* \models \text{irred}_{R_S}(S)$ , where  $R_S^*$  is the equality Herbrand interpretation defined by the congruence  $\overset{*}{\leftrightarrow}_{R_S}$  and  $\text{irred}_{R_S}(S)$  is the set of ground instances  $C\sigma$  of clauses  $C \mid T$  in  $S$  such that  $\sigma \models T$  and  $\sigma$  is irreducible by  $R_S$  ( $\sigma$  is *irreducible* by  $R_S$  if  $x\sigma$  is irreducible for all  $x \in \text{Dom}(\sigma)$ ); furthermore, we denote by  $\text{nf}_{R_S}(\sigma)$  the substitution  $\sigma'$  such that  $x\sigma' = \text{nf}_{R_S}(x\sigma)$  for all  $x \in \text{Dom}(\sigma)$ .

We proceed by induction on  $(\succ_R)_c$ , that is, we derive a contradiction from the existence of a minimal (w.r.t.  $(\succ_R)_c$ ) ground instance  $C\sigma \in \text{irred}_{R_S}(S)$  for some  $C \mid T \in S$ , where  $\sigma \models T$ , such that  $R_S^* \not\models C\sigma$ . In the following, we (ambiguously) write  $\succ_R$  for terms, equations and clauses instead of  $\succ_R$ ,  $(\succ_R)_e$  and  $(\succ_R)_c$  respectively:

1. If  $C$  is of the form  $\rightarrow s \simeq t$  then we assume, w.l.o.g., that  $C\sigma$  is of the form  $\rightarrow s\sigma \simeq t\sigma$  with  $s\sigma \succ t\sigma$ . Since  $R_S^* \not\models C\sigma$ , we have that  $s\sigma \rightarrow t\sigma \notin R_S$ , i.e.  $C\sigma$  has not generated the rule  $s\sigma \rightarrow t\sigma$  due to one of the following reasons: either  $s\sigma$  or  $t\sigma$  are reducible by  $R_C$ . Let us consider that  $s\sigma$  is reducible (the proof for  $t\sigma$  is analogous). Assume  $s\sigma$  is reducible by some rule  $l\sigma \rightarrow r\sigma \in R_S$ , and  $l\sigma \rightarrow r\sigma$  has been generated by an instance  $C'\sigma$  of some clause  $C' \mid T'$  in  $S$ , where  $C'$  is of the form  $\rightarrow l \simeq r$  with  $l\sigma \succ r\sigma$ . Now we

have  $s\sigma|_p \equiv l\sigma$  and, since  $\sigma$  is irreducible by  $R_S$ , the only possibility is now that  $p$  is a non-variable position of  $s$ . (Note that, since we assume that there are no name clashes between the—different—variables of  $C$  and  $C'$ , we can consider that the instances of  $C$  and of  $C'$  under consideration are both by the same ground  $\sigma$ .) Then there exists an inference by paramodulation right:

$$\frac{\rightarrow l \simeq r \mid T' \quad \rightarrow s \simeq t \mid T}{\rightarrow s[r]_p \simeq t \mid s|_p = l \wedge l > r \wedge T' \wedge T}$$

whose conclusion  $D$  has an instance  $D\sigma$  where  $\sigma \models s|_p = l \wedge l > r \wedge T \wedge T'$  such that  $C\sigma \succ_R D\sigma$  and where  $R_S^* \not\models D\sigma$ . Furthermore,  $D\sigma \in \text{irred}_{R_S}(S)$ : indeed  $x\sigma$  is irreducible by  $R_S$  for all variables  $x \in \text{vars}(D)$ . This is clearly the case if  $x \in \text{vars}(C)$ . For  $x \in \text{vars}(C')$ , there are two possibilities: if  $x \equiv l$  then  $x \notin \text{vars}(D)$  since  $l\sigma \succ r\sigma$  and  $\succ \supset \triangleleft$ ; if  $x \not\equiv l$  then  $x\sigma$  is irreducible w.r.t.  $R_{C'}$  by construction of  $R_S$ , and also w.r.t.  $R_S$ , since for all rules  $l' \leftarrow r' \in R_S \setminus R_{C'}$  we have  $l' \succeq l\sigma \succ x\sigma$  and hence such rules cannot reduce  $x\sigma$ . Altogether, this contradicts the minimality of  $C\sigma$ .

**2.** If  $C$  is of the form  $\Gamma, s \simeq t \rightarrow \Delta$ , from the fact that  $R_S^* \not\models C\sigma$  it necessarily follows that  $s\sigma \simeq t\sigma \in R_S^*$  and, since  $R_S$  is convergent,  $s\sigma$  and  $t\sigma$  must have the same normal form w.r.t.  $R_S$ , i.e. there must exist a term  $v$  such that  $s\sigma \xrightarrow{*}_{R_S} v \xleftarrow{*}_{R_S} t\sigma$ . The following cases have to be considered:

**2a.**  $s\sigma \equiv t\sigma$ . In this case there exists an inference by equality resolution:

$$\frac{\Gamma, s \simeq t \rightarrow \Delta \mid T}{\Gamma \rightarrow \Delta \mid s = t \wedge T}$$

whose conclusion  $D$  has an instance  $D\sigma$  with the same but less terms than  $C\sigma$ , and hence  $C\sigma \succ_R D\sigma$ . Since  $R_S^* \not\models C\sigma$  we also have  $R_S^* \not\models D\sigma$ . This contradicts the minimality of  $C\sigma$ .

**2b.**  $s\sigma \not\equiv t\sigma$  and  $s\sigma \xrightarrow{+}_{R_S} v \xleftarrow{*}_{R_S} t\sigma$ . In this case  $s\sigma$  is reducible at some non-variable position  $p$  by some rule  $l\sigma \rightarrow r\sigma$ , which has been generated by an instance  $C'\sigma$  of some clause  $C' \mid T'$  in  $S$ , where  $C'$  is of the form  $\rightarrow l \simeq r$  with  $l\sigma \succ r\sigma$ . Then there exists an inference by paramodulation left:

$$\frac{\rightarrow l \simeq r \mid T' \quad \Gamma, s \simeq t \rightarrow \Delta \mid T}{\Gamma, s[r]_p \simeq t \rightarrow \Delta \mid s|_p = l \wedge l > r \wedge T' \wedge T}$$

whose conclusion  $D$  has an instance  $D\sigma$  such that  $C\sigma \succ_R D\sigma$  and  $R_S^* \not\models D\sigma$ . Furthermore,  $D\sigma \in \text{irred}_{R_S}(S)$ : for all variables  $x \in \text{vars}(D)$  such that  $x \in \text{vars}(C)$  we have that  $x\sigma$  is irreducible by  $R_S$ ; for all variables  $x \in \text{vars}(C')$  we have  $x \equiv l$  or  $x \not\equiv l$ , and the two situations are solved in the same way as before. Altogether, this contradicts the minimality of  $C\sigma$ .

**2c.**  $s\sigma \not\equiv t\sigma$  and  $s\sigma \xleftarrow{+}_{R_S} t\sigma$ . This case is analogous to the previous one.

Once we have  $R_S^* \models \text{irred}_{R_S}(S)$ , then also  $R_S^* \models \text{irred}_{R_S}(S_0)$  since  $S \supseteq S_0$ . But then  $R_S^* \models S_0$  as well, since for each ground instance  $C\sigma$  of a clause in  $S_0$ , if  $\sigma' = \text{nf}_{R_S}(\sigma)$  then  $C\sigma'$  is an existing instance of a clause in  $\text{irred}_{R_S}(S_0)$  as clauses in  $S_0$  have no constraints, and then  $R_S^* \models C\sigma'$ , which clearly implies  $R_S^* \models C\sigma$ . Finally, since  $S_0 \models S$ , from  $R_S^* \models S_0$  we obtain  $R_S^* \models S$ .  $\square$

**Definition 11** *The inference system  $\mathcal{LM}$  (for left-maximal) is the particular case of the inference system  $\mathcal{J}$  where always the selected negative equation is maximal w.r.t.  $\succ_e$  in the antecedent.*

**Corollary 12**  *$\mathcal{LM}$  w.r.t. any west-ordering  $\succ$  is refutation complete.*

### 3 Selection of maximal equations

In this section we will prove the refutation completeness of the following inference system  $\mathcal{M}$  on constrained clauses. Its main difference with respect to  $\mathcal{J}$  and  $\mathcal{LM}$  is that it does not require that in each clause a negative equation is selected whenever there is any. Instead, inferences with  $\mathcal{M}$  only involve maximal equations ( $\mathcal{M}$  stands for maximal), even if they are positive and the antecedent is non-empty. Hence  $\mathcal{M}$  does not necessarily lead to positive unit strategies, like it happened with the former two inference systems.

**Definition 13** *The rules of the inference system  $\mathcal{M}$  are as follows:*

paramodulation right:

$$\frac{\Gamma' \rightarrow l \simeq r \mid T_1 \quad \Gamma \rightarrow s \simeq t \mid T_2}{\Gamma', \Gamma \rightarrow s[r]_p \simeq t \mid s|_p = l \wedge l > r \wedge (l \simeq r) > \Gamma' \wedge (s \simeq t) > \Gamma \wedge T_1 \wedge T_2} \quad \text{if } s|_p \notin \mathcal{X}$$

paramodulation left:

$$\frac{\Gamma' \rightarrow l \simeq r \mid T_1 \quad \Gamma, s \simeq t \rightarrow \Delta \mid T_2}{\Gamma', \Gamma, s[r]_p \simeq t \rightarrow \Delta \mid s|_p = l \wedge l > r \wedge (l \simeq r) > \Gamma' \wedge (s \simeq t) \geq \Gamma, \Delta \wedge T_1 \wedge T_2} \quad \text{if } s|_p \notin \mathcal{X}$$

equality resolution:

$$\frac{\Gamma, s \simeq t \rightarrow \Delta \mid T}{\Gamma \rightarrow \Delta \mid s = t \wedge (s \simeq t) \geq \Gamma, \Delta \wedge T}$$

In order to prove the completeness of  $\mathcal{M}$  we will proceed as follows. Assume  $S$  is a set of constrained clauses that is closed under  $\mathcal{M}$ . Furthermore, let  $P$  be a proof by  $\mathcal{LM}$  deriving the empty clause from  $S$ . Then we will show that if  $P$  is non-trivial, i.e., it has more than zero steps, then there exists another proof by  $\mathcal{LM}$  from  $S$  of the empty clause with a smaller number of steps. By induction on this proof transformation process, it follows that the empty clause belongs to  $S$ .

Let  $S$  be a set of constrained clauses and let  $C \mid T$  be a constrained clause that is in the closure of  $S$  w.r.t.  $\mathcal{LM}$ . Then, as usual, the proof by  $\mathcal{LM}$  of  $C \mid T$  from  $S$  can be expressed as a tree rooted by  $C \mid T$ , and whose leaves are in  $S$ . Now assume  $T$  is satisfiable, and let  $\sigma$  be a ground solution of  $T$ . Furthermore,  $\sigma$  can be taken such that its domain contains all variables occurring in  $S$  and in the proof. Therefore we can deal with ground proofs where the constraints are replaced by their solution  $\sigma$ : by a (ground)  $\mathcal{LM}$ -proof  $P$  of  $C \mid \sigma$  from  $S$  we mean a proof tree by  $\mathcal{LM}$ , whose nodes are clauses of the form  $D \mid \sigma$ , and whose leaves are clauses  $D' \mid \sigma$  where  $D' \mid T'$  is in  $S$  and  $\sigma \models T'$ .

By  $\text{steps}(P)$  we refer to its number of proof steps (or, equivalently, to its number of non-leaf nodes).

The following is an example of an  $\mathcal{LM}$ -proof:

$$\frac{\frac{\rightarrow c \simeq a \mid x \mapsto a}{\rightarrow b \simeq a \mid x \mapsto a} \quad \frac{x \simeq a \rightarrow b \simeq c \mid x \mapsto a}{\rightarrow b \simeq c \mid x \mapsto a}}{\rightarrow b \simeq a \mid x \mapsto a}$$

When dealing with  $\mathcal{LM}$ -proofs  $P$ , we will frequently speak about its rightmost leaf ( $x \simeq a \rightarrow b \simeq c \mid x \mapsto a$  in the example), its rightmost inner node ( $\rightarrow b \simeq c \mid x \mapsto a$ ), its rightmost step (the inference obtaining  $\rightarrow b \simeq c \mid x \mapsto a$  from  $x \simeq a \rightarrow b \simeq c \mid x \mapsto a$ ), and its rightmost path (the nodes  $x \simeq a \rightarrow b \simeq c \mid x \mapsto a$ ,  $\rightarrow b \simeq c \mid x \mapsto a$ ,  $\rightarrow b \simeq a \mid x \mapsto a$ ). An  $\mathcal{LM}$ -proof  $P$  is called *antecedent elimination of*  $\Gamma$  if its rightmost leaf is of the form  $\Gamma \rightarrow \Delta \mid \sigma$ , its root is  $\rightarrow \Delta \mid \sigma$ , and no node on its rightmost path is obtained by a paramodulation-right step.

Note that in antecedent eliminations it is irrelevant which is  $\Delta$ : for any  $\Delta'$ , the same proof (up to replacements of  $\Delta$  by  $\Delta'$ ) proves  $\rightarrow \Delta' \mid \sigma$  from a rightmost leaf  $\Gamma \rightarrow \Delta' \mid \sigma$ . Therefore in the antecedent eliminations of the following lemma we consider only empty succedents.

### 3.1 Completeness proof

**Lemma 14** (fusion lemma) *Let  $P_1$  and  $P_2$  be two antecedent elimination  $\mathcal{LM}$ -proofs that are with rightmost leaves  $\Gamma_1 \rightarrow \mid \sigma$ , and  $\Gamma_2 \rightarrow \mid \sigma$  respectively.*

*Then there exists an antecedent elimination  $\mathcal{LM}$ -proof  $P$  with rightmost leaf  $\Gamma_1, \Gamma_2 \rightarrow \mid \sigma$  such that  $\text{steps}(P) = \text{steps}(P_1) + \text{steps}(P_2)$ , and every non-rightmost leaf of  $P$  is a non-rightmost leaf of  $P_1$  or of  $P_2$ .*

**Proof:** If  $\Gamma_1$  is empty or  $\Gamma_2$  is empty, the proof is trivial. For the remaining cases, we proceed by induction on  $\text{steps}(P_1) + \text{steps}(P_2)$ . Let  $e$  be the equation in  $\Gamma_1 \cup \Gamma_2$  such that the rightmost step in  $P_1$  or  $P_2$  takes place on  $e$ , and such that  $e\sigma$  is maximal w.r.t.  $\succ_e$  in  $\Gamma_1\sigma \cup \Gamma_2\sigma$ . Note that such an  $e$  must exist. Wlog. assume  $e$  is in  $\Gamma_1$ , that is,  $\Gamma_1$  is of the form  $\Gamma'_1, e$ .

Now the rightmost step of  $P_1$  can be by equality resolution or by paramodulation-left.

1. If it is by equality resolution, then it is of the form:

$$\frac{\Gamma, e \rightarrow \mid \sigma}{\Gamma \rightarrow \mid \sigma}$$

Let  $P'_1$  be like  $P_1$ , but where the rightmost leaf is removed, i.e., where the conclusion of the rightmost step of  $P_1$  is already a leaf. Furthermore, let the rightmost leaf of  $P_2$  be of the form  $\Gamma_2 \rightarrow \Delta_2 \mid \sigma$ . Then we can apply the induction hypothesis on  $P'_1$  and  $P_2$ , getting a new proof  $P'$ , whose rightmost leaf is  $\Gamma'_1, \Gamma_2 \rightarrow \mid \sigma$ .

Now let  $P$  be the  $\mathcal{LM}$ -proof formed by  $P'$ , and where above its rightmost leaf we insert the inference by  $\mathcal{LM}$

$$\frac{\Gamma'_1, e, \Gamma_2 \rightarrow \sigma}{\Gamma'_1, \Gamma_2 \rightarrow \mid \sigma}$$

The requirements about the number of steps are met:  $\text{steps}(P) = \text{steps}(P') + 1$ , where, by induction hypothesis,  $\text{steps}(P') = (\text{steps}(P_1) - 1) + \text{steps}(P_2)$ , which altogether implies  $\text{steps}(P) = \text{steps}(P_1) + \text{steps}(P_2)$ .



Finally, every non-rightmost leaf of  $P$  is a non-rightmost leaf of  $P'$ , which, by induction hypothesis, is a non-rightmost leaf of  $P'_1$  (and then of  $P_1$ ) or of  $P_2$ .

**2.** If the rightmost step of  $P_1$  is by paramodulation-left on the equation  $e$  of its rightmost leaf  $\Gamma'_1, e \rightarrow \mid \sigma$ , then it is a step of the form:

$$\frac{\rightarrow \Delta_{11} \mid \sigma \quad \Gamma'_1, e \rightarrow \mid \sigma}{\Gamma'_1, e' \rightarrow \mid \sigma}$$

where  $e'$  is the result of the paramodulation on  $e$ . Let  $P_{11}$  be the subproof of  $P_1$  rooted by  $\rightarrow \Delta_{11} \mid \sigma$ . Now let  $P'_1$  be like  $P_1$ , but where the subproof  $P_{11}$  and the rightmost leaf are removed, i.e., where the conclusion of the rightmost step of  $P_1$  is already a leaf. Then we can apply the induction hypothesis on  $P'_1$  and  $P_2$ , getting a new proof  $P'$ , whose rightmost leaf is  $\Gamma'_1, e', \Gamma_2 \rightarrow \mid \sigma$ .

Now let  $P$  be the  $\mathcal{LM}$ -proof formed by  $P'$ , and where above its rightmost leaf we insert the inference by  $\mathcal{LM}$

$$\frac{\rightarrow \Delta_{11} \mid \sigma \quad \Gamma'_1, e, \Gamma_2 \rightarrow \mid \sigma}{\Gamma'_1, e', \Gamma_2 \rightarrow \mid \sigma}$$

and where above  $\rightarrow \Delta_{11} \mid \sigma$  we insert the subproof  $P_{11}$ .

The requirements about the number of steps are met, since  $steps(P) = steps(P_{11}) + steps(P') + 1$ , where, by induction hypothesis,  $steps(P') = (steps(P_1) - 1 - steps(P_{11}) + steps(P_2))$ , which altogether implies  $steps(P) = steps(P_1) + steps(P_2)$ .

Finally, every non-rightmost leaf of  $P$  is either a leaf of  $P_{11}$  or a non-rightmost leaf of  $P'$ . Every leaf of  $P_{11}$  is a non-rightmost leaf of  $P_1$ . Moreover, by induction hypothesis, every non-rightmost leaf of  $P'$  is a non-rightmost leaf of  $P'_1$  (and then of  $P_1$ ) or of  $P_2$ . Altogether implies that every non-rightmost leaf of  $P$  is a non-rightmost leaf of  $P_1$  or of  $P_2$ .  $\square$

In the following, the  $\mathcal{LM}$ -proof  $P$  built as in the previous lemma will be called the *fusion* of  $P_1$  and  $P_2$ .

**Lemma 15** (*General fusion lemma*) *Let  $P_1$  be an  $\mathcal{LM}$ -proof such that is antecedent elimination of  $\Gamma_1$ . Let  $P_2$  be an  $\mathcal{LM}$ -proof with rightmost leaf  $\Gamma_2 \rightarrow \Delta_2 \mid \sigma$  such that a subtree in its rightmost path is antecedent elimination (i.e. the antecedent  $\Gamma_2$  is eliminated in  $P_2$ , but after this elimination, some right paramodulation inferences can be made on the rightmost path).*

*Then there exists an  $\mathcal{LM}$ -proof  $P$  such that its rightmost leaf is  $\Gamma_1, \Gamma_2 \rightarrow \Delta_2 \mid \sigma$ , and the antecedent is eliminated on the rightmost path. Moreover  $steps(P) = steps(P_1) + steps(P_2)$ , and every non-rightmost leaf of  $P$  is a non-rightmost leaf of  $P_1$  or of  $P_2$ .*

**Proof:** Let  $P_{21}$  be the antecedent elimination  $\mathcal{LM}$ -proof on the rightmost path of  $P_2$ . Let  $P'_2$  be like  $P_2$  but where  $P_{21}$  is removed. Note that the rightmost leaf of  $P'_2$  is  $\rightarrow \Delta_2 \mid \sigma$ , and  $P_2$  can be obtained by inserting  $P_{21}$  on the rightmost path of  $P'_2$ . Moreover  $steps(P_2) = steps(P'_2) + steps(P_{21})$ . Since  $P_1$  and  $P_{21}$  are antecedent elimination, by the fusion lemma, there exists  $P'$  such that it is antecedent elimination of  $\Gamma_1 \cup \Gamma_2$ ,  $steps(P') = steps(P_1) + steps(P_{21})$  and all the non-rightmost leaves of  $P'$  are non-rightmost leaves of  $P_1$  or of  $P_{21}$  (and then of  $P_2$ ). Moreover, the root of  $P'$  can be chosen to be

$\rightarrow \Delta_2$ , maintaining these conditions. Then, the rightmost leave of  $P'$  is  $\Gamma_1, \Gamma_2 \rightarrow \Delta_2 \mid \sigma$ . Now let  $P$  be the  $\mathcal{LM}$ -proof formed by inserting  $P'$  on the rightmost leaf of  $P'_2$ . Then,  $P$  satisfy all the conditions we are looking for.  $\square$

**Lemma 16** *Let  $S$  be a set of clauses closed by  $\mathcal{M}$ , and  $P$  an  $\mathcal{LM}$ -proof of  $\rightarrow \Delta \mid \sigma$  from  $S$ .*

*Then there is an antecedent elimination  $\mathcal{LM}$ -proof  $P'$  from  $S$  of  $\rightarrow \Delta \mid \sigma$  whose rightmost leaf is of the form  $\Gamma \rightarrow \Delta \mid \sigma$ , where  $\text{steps}(P') \leq \text{steps}(P)$  and  $\Delta\sigma(\succ_e)_{mul}\{e\}$  for all  $e$  in  $\Gamma\sigma$  (hence if  $\Delta$  is empty,  $\Gamma$  is empty as well).*

**Proof:** We will proceed by induction on  $\text{steps}(P)$ . In this proof the substitution part  $\mid \sigma$  of the clauses is omitted in order to improve readability.

Let the rightmost leaf of  $P$  be of the form  $\Gamma_1 \rightarrow \Delta_1$ . There are several cases to be considered:

- 0. If  $\Gamma_1$  and  $\Delta_1$  are both empty, then  $P$  has no steps and  $P'$  can be  $P$  itself.
- 1. The maximal equation of  $\Gamma_1\sigma \rightarrow \Delta_1\sigma$  is in  $\Delta_1\sigma$ . We consider two possibilities depending on whether some right paramodulation inference is made or not on the rightmost path.
  - 1a. In the case that no right paramodulation inference is made on the rightmost path, the  $P'$  we are looking for, is directly  $P$ , and then  $\Gamma$  corresponds to  $\Gamma_1$ .
  - 1b. Suppose now there are some right paramodulation inferences on the rightmost path of  $P$ . Then the highest one is of the form:

$$\frac{\frac{\rightarrow \Delta_2 \quad \rightarrow \Delta_1}{\rightarrow \Delta_3}}{\rightarrow \Delta_3}$$

Let  $P_1$  be the subproof rooted by  $\rightarrow \Delta_1$ . It is an antecedent elimination. Let  $P_2$  be the subproof rooted by  $\rightarrow \Delta_2$ . Let  $P_3$  be like  $P$ , but where  $P_1$  and  $P_2$  are removed (i.e.  $\rightarrow \Delta_3$  is the rightmost leaf of  $P$ ). Then we have  $\text{steps}(P) = \text{steps}(P_1) + \text{steps}(P_2) + \text{steps}(P_3) + 1$ . We apply the induction hypothesis on  $P_2$ , and there exists an antecedent elimination  $\mathcal{LM}$ -proof  $P'_2$  of  $\rightarrow \Delta_2$  from  $S$  such that its rightmost leaf is of the form  $\Gamma_2 \rightarrow \Delta_2$ , where  $\Delta_2\sigma > \Gamma_2\sigma$  and  $\text{steps}(P'_2) \leq \text{steps}(P_2)$ . If we apply the fusion lemma to  $P'_2$  and  $P_1$ , we obtain an antecedent elimination  $\mathcal{LM}$ -proof  $P_4$  of  $\rightarrow \Delta_3$  from  $S$ , where its rightmost leaf is  $\Gamma_1, \Gamma_2 \rightarrow \Delta_3$ , and  $\text{steps}(P_4) = \text{steps}(P'_2) + \text{steps}(P_1) \leq \text{steps}(P_2) + \text{steps}(P_1)$ . Now let  $P_5$  be the  $\mathcal{LM}$ -proof formed by  $P_3$ , and where above its rightmost leaf we insert  $P_4$ . We have  $\rightarrow \Delta$  in the root of  $P_5$ , and  $\text{steps}(P_5) = \text{steps}(P_3) + \text{steps}(P_4) \leq \text{steps}(P_3) + \text{steps}(P_2) + \text{steps}(P_1)$ . Moreover, every non-leaf of  $P_5$  is from  $S$ : all the non-rightmost leaves of  $P_3$  are from  $S$ , and, since  $P_4$  is the fusion of  $P'_2$  and  $P_1$ , then, its non-rightmost leaves are from  $S$  too. But also, the rightmost leave of  $P_5$  is from  $S$ , since the next inference is an  $\mathcal{M}$ -inference from  $S$ :

$$\frac{\Gamma_2 \rightarrow \Delta_2 \quad \Gamma_1 \rightarrow \Delta_1}{\Gamma_2, \Gamma_1 \rightarrow \Delta_3}$$

Then the  $\mathcal{LM}$ -proof  $P'$  we are looking for, is the one obtained by applying the induction hypothesis on  $P_5$ .

**2.** The maximal equation of  $\Gamma_1\sigma \rightarrow \Delta_1\sigma$  is in  $\Gamma_1\sigma$ . Then,  $\Gamma_1$  is of the form  $\Gamma_{11}, e$  and the rightmost step of  $P$  is an equality resolution or a left paramodulation inference on  $e$ :

**2a.** If it is an equality resolution step:

$$\frac{\Gamma_{11}, e \rightarrow \Delta_1}{\Gamma_{11} \rightarrow \Delta_1}$$

then this  $\mathcal{LM}$ -inference is also an  $\mathcal{M}$ -inference. Therefore  $\Gamma_{11} \rightarrow \Delta_1$  is in  $S$ . Let  $P_3$  be like  $P$  but where the rightmost leaf has been removed. All leaves of  $P_3$  are from  $S$ , and  $steps(P_3) < steps(P)$ . Then the  $\mathcal{LM}$ -proof  $P'$  we are looking for, is the one obtained by applying the induction hypothesis on  $P_3$ .

**2b.** If the rightmost step of  $P$  is a left paramodulation inference, it is of the form:

$$\frac{\rightarrow \Delta_2 \quad \Gamma_{11}, e \rightarrow \Delta_1}{\Gamma_{11}, e' \rightarrow \Delta_1}$$

Let  $P_2$  be the subproof of  $P$  rooted by  $\rightarrow \Delta_2$ . Let  $P_3$  be like  $P$  but where the subproof  $P_2$  and the rightmost leaf are removed (the rightmost leaf of  $P_3$  is  $\Gamma_{11}, e' \rightarrow \Delta_1$ ). Note that all the non-rightmost leaves of  $P_3$  are clauses from  $S$ . We have  $steps(P) = steps(P_2) + steps(P_3) + 1$ . We apply induction hypothesis on  $P_2$ , and there exists an antecedent elimination  $\mathcal{LM}$ -proof  $P'_2$  of  $\rightarrow \Delta_2$  from  $S$  such that its rightmost leaf is of the form  $\Gamma_2 \rightarrow \Delta_2$ , where  $\Delta_2\sigma > \Gamma_2\sigma$  and  $steps(P'_2) \leq steps(P_2)$ . Now, we apply the general fusion lemma to  $P'_2$  and  $P_3$ , and then, exists an  $\mathcal{LM}$ -proof  $P_4$  such that its root is  $\rightarrow \Delta$ , its rightmost leaf is  $\Gamma_{11}, e', \Gamma_2 \rightarrow \Delta_1$ ,  $steps(P_4) = steps(P_3) + steps(P'_2) < steps(P)$  and all the non-rightmost leaves of  $P_4$  are non-rightmost leaves of  $P_3$  or of  $P'_2$ , and then, from  $S$ . But we will see now that the rightmost leave of  $P_4$  is a clause from  $S$ , too, since the next inference is an  $\mathcal{M}$ -inference from  $S$ :

$$\frac{\Gamma_2 \rightarrow \Delta_2 \quad \Gamma_{11}, e \rightarrow \Delta_1}{\Gamma_2, \Gamma_{11}, e' \rightarrow \Delta_1}$$

Then the  $\mathcal{LM}$ -proof  $P'$  we are looking for is the one obtained by applying the induction hypothesis on  $P_4$ . □

**Theorem 17** (*Completeness theorem*) *Let  $S_0$  be an unconstrained set of clauses, and  $S$  be the closure by  $\mathcal{M}$  of  $S$ . If  $S_0$  is unsatisfiable, then  $\square \in S$ .*

**Proof:** By completeness of  $\mathcal{LM}$  there is an  $\mathcal{LM}$ -proof of  $\square$  from  $S$ . Then, applying lemma 16 to the case where  $\Delta$  is empty gives us a trivial  $\mathcal{LM}$ -proof of  $\square$ . Consequently  $\square \in S$ . □

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